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1994 J. Phys. A: Math. Gen. 27 1645

(http://iopscience.iop.org/0305-4470/27/5/028)

J. Phys. A: Math. Gen. 27 (1994) 1645–1664. Printed in the UK

On integrability of the inhomogeneous Heisenberg ferromagnet model: examination of a new test

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Received 22 March 1993

Abstract. We present an improved version of the integrability test based on application of Lie point transformations acting on the bare 'linear problem'. The test is then applied to the inhomogeneous Heisenberg ferromagnet model equation. The integrable cases selected by this test are identical with those obtained by the standard Painlevé test.

1. Introduction

The integrable systems of nonlinear partial differential equations ('soliton systems') have the important property of being equivalent to the integrability conditions for some linear system of partial differential equations *with* a parameter (the so-called spectral parameter). This linear system is known as a Lax pair (or linear problem) associated with a given integrable system [1, 2].

On the other hand, numerous nonlinear systems are equivalent to the integrability conditions for some linear system *without* any parameter (non-parametric 'Lax pair' or bare 'linear problem'). The so-called Gauss–Mainardi–Codazzi–Ricci equations (see for example [3–5]) describing immersions of surfaces into ambient spaces form a very large class of systems of that kind. To isolate integrable systems within such a class one has (at least) to insert a 'good' parameter into the corresponding non-parametric 'Lax pair'.

It seems natural to do that by an appropriate one-parameter group of transformations which *leaves unchanged* the considered nonlinear system, and, at the same time, *changes* the linear problem. Apparently, it results in an introduction of the group parameter into this linear problem.

The possibility of an identification of the spectral parameter with a group parameter was noticed many years ago (see for instance [6, 7]) but the considered transformations were quite simple (scaling, Galilean boosts, etc). The systematic approach to that problem was first given by Levi *et al* [8]. In particular, they proposed the study of all one-parameter groups of Lie point symmetries. An extension of the approach of [8] to non-local transformations was considered in [9, 10].

There is another, trivial, possibility of inserting a parameter: by application of any parameter-dependent gauge transformation. It is commonly believed, however, that a 'good' spectral parameter cannot be gauged out. Therefore transformations inserting a parameter in this trivial way should be considered as the symmetries of the non-parametric 'Lax pair'.

The above considerations yield a working algorithm to isolate integrable systems (see [11, 12]). It consists of the following steps:

(1) computing the algebra of infinitesimal Lie point symmetries of the nonlinear system;
(2) computing the algebra of infinitesimal Lie point symmetries of the corresponding non-parametric 'Lax pair';
(3) comparison between Lie algebras obtained this way: if they are not identical then the nonlinear system under consideration is conjectured to be integrable;
(4) calculation of the one-parameter group corresponding to any vector field which generates a symmetry of the nonlinear system but does not generate a symmetry of the non-parametric 'Lax pair': the action of this group inserts the parameter which is supposed to be a 'good' spectral parameter. In principle, the obtained *parametric* Lax pair can be further used to prove the expected integrability.

The test described above works properly when applied to some particular cases including the so-called 'AKNS class' of soliton systems ([13], see also [7]).

One can also hope to identify some new integrable systems this way. The first case for which such a procedure turned out to be successful is the so-called Bianchi system for which a non-isospectral linear problem was obtained [14].

The aim of this paper is to perform one more test of the above described algorithm; namely, we discuss in detail the application of the above algorithm to the inhomogeneous Heisenberg ferromagnet equation (1).

The paper is organized as follows. In section 2 we discuss the inhomogeneous Heisenberg ferromagnet equation. In sections 3 and 4 we apply the above algorithm based on Lie point symmetries. It turns out that *not all* integrable cases can be obtained this way. Then, in sections 5 and 6, we present an improved version of this algorithm: a more general class of symmetries ('extended' Lie point symmetries) is admitted. Finally, in section 7, we apply the standard Painlevé test to verify and confirm our conclusions.

All calculations and proofs have been placed in three appendices.

2. The inhomogeneous Heisenberg ferromagnet (IHF) equation and the non-homogeneous, nonlinear Schrödinger (NHNS) system

The dynamics of the one-dimensional classical inhomogeneous Heisenberg ferromagnet in the continuum limit is described by the following equation

$$\boldsymbol{S}_{,t}=\boldsymbol{S}\wedge(f\boldsymbol{S}_{,x})_{,x} \qquad \boldsymbol{S}^2=1 \tag{1}$$

where the unknown $\boldsymbol{S}=\boldsymbol{S}(x,t)\in\mathbb{E}^3$ is a unit vector function of one-dimensional space variable x and time t, while $f=f(x,t)\in\mathbb{R}$ is given. Physically, the function f—called the *coupling function*—is the continuum limit of the coupling 'constants' between neighbouring spins. Here we adopt the convention: comma stands for differentiation and $\wedge$ means cross-product.

The first geometrical interpretation of equation (1), based on the Lamb's approach [15, 16], was given by Lakshmanan and Bullough [17] for f linear in x and by Balakrishnan [18] for general f. Another geometrical interpretation of this model (see [19, 20]) is based on the so-called soliton surfaces approach [5].

The IHF equation (1) is equivalent (see [18, 19]) to the so-called non-homogeneous, nonlinear Schrödinger system

$$\mathrm{i}q_{,t}+(fq)_{,xx}+2qR=0 \tag{2a}$$

$$R_{,x}-(f|q|^2)_{,x}-f_{,x}|q|^2=0 \tag{2b}$$

where $q=q(x,t)\in\mathbb{C}$ and $R=R(x,t)\in\mathbb{R}$ are unknowns.

Indeed, some simple geometrical consideration (see [8]) lead to the following bare 'linear problem' associated with the system (2):

$$\Psi_{,x}=\begin{pmatrix}0 & q\\ -\bar{q} & 0\end{pmatrix}\Psi \qquad \Psi_{,t}=\begin{pmatrix}\mathrm{i}R & \mathrm{i}(fq)_{,x}\\ \mathrm{i}(f\bar{q})_{,x} & -\mathrm{i}R\end{pmatrix}\Psi \tag{3}$$

which is already known [18]. Taking into account the isomorphism $su(2)\cong\mathbb{E}^3$ (the commutator is identified with the vector product, etc) one can easily check that

$$S=\tfrac{1}{2}\Psi^{-1}\mathrm{i}\sigma_3\Psi \tag{4}$$

where $\sigma_3=\mathrm{diag}(1,-1)$, solves the IHF equation (1) for *any* f (this result was known earlier for the cases $f=$ constant [21] and $f=ax+b$ [22]).

Equation (1) for f linear in x is well known to be integrable [17]. Lakshmanan and Ganesan [22] formulated the inverse method and found the one-soliton solution in this case. The integrability of (2) for linear f was proved even earlier [23].

It has been proved in [24] that f linear in x is the only inhomogeneity function for which the system (2) has the Painlevé property—the usual companion to complete integrability (see section 7). A proof that linearity of f is both the necessary and sufficient condition for that property is given in appendix 3.

3. Introduction of the spectral parameter by Lie point symmetries

Let us consider a nonlinear system (denoted by $\Delta=0$) for m unknown functions $q^\alpha=q^\alpha(x^1,x^2)(\alpha=1,\ldots,m)$ which is equivalent to integrability conditions for a system of two linear equations of the following form:

$$\Psi_{,k}=U_k\Psi \qquad k=1,2 \tag{5}$$

where unknown $\Psi=\Psi(x^1,x^2)$ is a matrix function and matrices U_k depend on x^k, q^α and derivatives of q^α. The system (5) (bare 'linear problem') is denoted by $\Delta'=0$.

The algebra $\mathscr{A}$ of Lie point symmetries of the nonlinear system $\Delta=0$ is defined in the usual way [25]:

$$\mathscr{A}:=\{w:\mathrm{pr}^{(n)}w(\Delta)=0|_{\Delta=0}\} \tag{6a}$$

where $\mathrm{pr}^{(n)}w$ is the so-called nth prolongation of the vector field w

$$w=\xi^k(x,q)\partial_k+\eta^\alpha(x,q)\partial_\alpha. \tag{6b}$$

In (6*a*) n stands for the order of the system $\Delta=0$, while in (6*b*) ξ^k $(k=1,2)$ and η^α $(\alpha=1,\ldots,m)$ are scalar functions satisfying the so-called 'determining equations'. Obviously $\partial_k \equiv \partial/\partial x^k$, $\partial_\alpha \equiv \partial/\partial q^\alpha$ and the Einstein summation convention is assumed.

Similarly, the algebra $\mathscr{A}'$ of Lie point symmetries of the bare 'linear problem' is defined as follows [11, 12]:

$$\mathscr{A}' := \{v : v=\pi(V) \;\&\; \mathrm{pr}^{(n)} V(\Delta')=0|_{\Delta=0,\Delta'=0}\} \tag{7a}$$

where V is a vector field of the form

$$V=\Xi^k(x,q)\partial_k + H^\alpha(x,q)\partial_\alpha + M(x,q)\Psi\partial_\Psi \tag{7b}$$

π is the projection: $\pi(V)=\Xi^k\partial_k + H^\alpha\partial_\alpha$, Ξ^k $(k=1,2)$ and H^α $(\alpha=1,\ldots,m)$ are scalar functions, and M is a matrix function. In (7*b*) we have used ∂_Ψ to denote partial differentiation w.r. to all matrix elements of Ψ.

The determining equations which define the algebra $\mathscr{A}'$ can be reduced to the following form [11, 12]:

$$D_k(M)=[U_k,M]+\mathrm{pr}^{(n)}v(U_k)+D_k(\xi^j)U_j|_{\Delta=0} \tag{8}$$

where $k=1,2$ and D_k is the total derivative with respect to x^k.

We conjecture that the parameter inserted into the bare 'linear problem' by one-parameter group generated by any vector field $u\in\mathscr{A}-\mathscr{A}'$ (i.e. $u\in\mathscr{A}$ and $u\notin\mathscr{A}'$) is a 'good' spectral parameter.

4. Introduction of the spectral parameter by Lie point symmetries in the case of NHNS system

Applying the algorithm described in sections 1 and 3 to the NHNS system (2) and its bare 'linear problem' (3) we obtain (see appendix 1) that Lie algebras $\mathscr{A}$ and $\mathscr{A}'$ are different only in the following 2 cases:

(1) $$f=b(t) \tag{9a}$$

(2) $$f=a(t)x+K_1a(t)+K_2a(t)\int_0^t a(\tau)\,\mathrm{d}\tau \tag{9b}$$

where a and b are any functions of t [26].

In both cases:

$$\dim(\mathscr{A})-\dim(\mathscr{A}')=1. \tag{10}$$

The vector field belonging to $\mathscr{A}-\mathscr{A}'$ is given, respectively, by (see appendix 1):

(1) $$w_1=2\beta\partial_x+\mathrm{i}xq\partial_q \tag{11a}$$

(2) $$v_1=(2x+2K_1+K_2\alpha)\alpha\partial_x+\frac{\alpha^2}{a}\partial_t+(\mathrm{i}x-2\alpha)q\partial_q-\left(\frac{\alpha^2}{a}\right)^{\cdot}R\partial_R \tag{11b}$$

where the dot denotes the derivative with respect to t

$$\alpha=\int_0^t a(\tau)\,\mathrm{d}\tau \qquad \beta=\int_0^t b(\tau)\,\mathrm{d}\tau. \tag{12}$$

Of course, any element of $\mathscr{A}'$ may be added to w_1 or v_1.

One-parameter groups generated by vector fields (11) can be computed in the standard way. The group parameter will be denoted by k.

(1) w_1 generates the group $T_k \equiv \exp(kw_1)$ which acts as follows:

$$\begin{aligned} &T_k x = x + 2\beta k \qquad T_k q = q \exp(\mathrm{i}(kx + \beta k^2)) \\ &T_k t = t \qquad T_k R = R. \end{aligned} \tag{13}$$

(2) v_1 generates the group $T_k \equiv \exp(kv_1)$ given by

$$\alpha(T_k t) = \alpha(t)/S \qquad \text{where} \quad S_k := 1 - k\alpha(t) \tag{14a}$$

$$T_k x = (x + K_1 + \alpha K_2)/S_k^2 - \alpha K_2/S_k - K_1 \tag{14b}$$

$$T_k q = q S_k^2 \exp\left(\frac{\mathrm{i}}{S_k}(kx + K_2(k\alpha + S_k \log S_k) + K_1 k^2 \alpha)\right) \tag{14c}$$

$$T_k R = R/(T_k t)^{\cdot} \tag{14d}$$

and, as a consequence

$$f(T_k x, T_k t) = S_k^{-2}[a(T_k t)/a(t)] f(x, t). \tag{14e}$$

Transforming the bare 'linear problem' (3) according to the transformations (13) and (14) and then performing a gauge transformation given, respectively, by

$$(1) \qquad T_k \Psi = \exp\left(\frac{\mathrm{i}}{2}\sigma_3(kx + \beta k^2)\right)\Psi \tag{15a}$$

$$(2) \qquad T_k \Psi = \exp\left(\frac{\mathrm{i}\sigma_3}{2S_k}(kx + K_2(k\alpha + S_k \log S_k) + K_1 k^2 \alpha)\right)\Psi \tag{15b}$$

we obtain the following *parametric* linear problem:

$$\begin{aligned} \Psi_{,x} &= \begin{pmatrix} \mathrm{i}\lambda & q \\ -\bar{q} & -\mathrm{i}\lambda \end{pmatrix}\Psi \\ \Psi_{,t} &= \begin{pmatrix} \mathrm{i}R - 2\mathrm{i}f\lambda^2 & -2q\lambda f + \mathrm{i}(fq)_{,x} \\ \mathrm{i}(f\bar{q})_{,x} + 2\lambda\bar{q}f & -\mathrm{i}R + 2\mathrm{i}f\lambda^2 \end{pmatrix}\Psi \end{aligned} \tag{16}$$

where λ is given respectively by:

$$(1) \qquad \lambda = k/2 \tag{17a}$$

$$(2) \qquad \lambda = k/(2 + 2k\alpha). \tag{17b}$$

One can easily recognize that (16) is identical with the standard parametric linear problem associated with the NHNS system (see [17, 27, 28]).

Therefore in the case of the NHNS system, Lie point symmetries always introduce a 'good' spectral parameter (in agreement with our conjecture).

However, the same linear problem exists for *any* coupling function of the form $f = a(t)x + b(t)$. Moreover, the parameter λ is given by (17*b*) for arbitrary a and b.

Thus, unfortunately, in some integrable cases (those corresponding to f linear in x but different from (9)), inserting the spectral parameter by Lie *point* symmetries turns out to be impossible.

5. Extended Lie point symmetries

The important conclusion of section 4 is that the Lie point symmetries are not sufficient to isolate all integrable cases. To improve our test of integrability it is necessary to admit a more general class of symmetries.

First of all, let us consider transformations which change the coupling function: the function $f=f(x, t)$ is transformed into some other function $\tilde{f}=\tilde{f}(\tilde{x}, \tilde{t})$ about which we make no assumptions. To compute such symmetries we apply a procedure which is similar to the standard algorithm of [25] but gives a larger class of symmetries: 'extended' Lie point symmetries ([12]).

In general, this procedure can be applied to those systems of differential equations which contain some function F as a parameter.

Let $\Delta=0$ be a system of differential equations whose independent and dependent variables are denoted by $x=(x^1, x^2)$ and $q=(q^1, \ldots, q^m)$, respectively. Assume that this system is parameterized by the function $F=F(x)$.

Consider the vector fields of the form:

$$\tilde{v}=\tilde{\xi}^k(x, q)\partial_k+\tilde{\eta}^\alpha(x, q)\partial_\alpha+\Phi(x)\partial_F \tag{18}$$

where $\tilde{\xi}^k$ $(k=1, 2)$, $\tilde{\eta}^\alpha$ $(\alpha=1, \ldots, m)$ and Φ are some functions.

We define extended Lie point symmetries as follows. The prolongation $\mathrm{pr}^{(n)}\tilde{v}$ is computed in the standard way [25]: f is treated as one more dependent variable. The prolonged vector field acts on the equation $\Delta=0$ in the usual way.

Working with the determining equations, we make a non-standard step: we stop treating F as one more variable and begin to treat it as a function of x^1, x^2. Thus $\tilde{\xi}^k$, $\tilde{\eta}^\alpha$ and Φ, a solution to the determining equations, may depend on F in a functional way (e.g. through some integrals) similarly to the case of Lie point symmetries (see, for instance, formulas (11)).

The extended Lie point symmetries are, obviously, a generalization of the standard Lie point symmetries which can be obtained from the former by imposing the following constraint on Φ

$$\Phi=F_{,1}\tilde{\xi}^1+F_{,2}\tilde{\xi}^2. \tag{19}$$

In particular, one can consider the extended Lie point symmetries of the bare 'linear problem' $\Delta'=0$. Thus, in analogy to the case discussed in section 3, we obtain the algebras $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}'$ of extended point symmetries of systems $\Delta=0$ and $\Delta'=0$, respectively:

$$\tilde{\mathscr{A}}:=\{\tilde{v}: \mathrm{pr}^{(n)}\tilde{v}(\Delta)=0|_{\Delta=0}\} \tag{20a}$$

$$\tilde{\mathscr{A}}':=\{\tilde{v}: \tilde{v}=\pi(\tilde{V})\ \&\ \mathrm{pr}^{(n)}\tilde{V}(\Delta')=0|_{\Delta=0,\Delta'=0}\} \tag{20b}$$

where $\tilde{v}$ is of the form (18), $\tilde{V}$ has an additional component $M(x, q)\Psi\partial_\Psi$, π is the projection along this component (similarly to section 3) and in the determining equations F is treated as a function of x rather than a variable.

6. Extended Lie point symmetries as a tool to isolate all known integrable cases of the NHNS system

Upon computing the algebras of extended Lie point symmetries for the NHNS system (2) and for its bare 'linear problem' (3) we obtain (see appendix 2) that $\tilde{\mathscr{A}}$ is identical to $\tilde{\mathscr{A}}'$ iff f is not linear in x.

The algebras $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}'$ are different iff $f = a(t)x + b(t)$, where a and b are any functions of t. Moreover:

$$\dim(\tilde{\mathscr{A}}) - \dim(\tilde{\mathscr{A}}') = 1 \tag{21}$$

and the following vector field belongs to $\tilde{\mathscr{A}} - \tilde{\mathscr{A}}'$:

$$u = 2(x\alpha + \beta)\partial_x + (\mathrm{i}x - 2\alpha)q\partial_q + 2a\alpha\partial_a + (4b\alpha - 2a\beta)\partial_b \tag{22}$$

where α and β are *non-local* variables defined by (12).

The integration of the vector field u, carried out in appendix 2, is non-trivial. As a result we obtain the following one-parameter family of transformations (computed earlier in a different way, see [9, 10])

$$T_k x = \frac{x}{(1-k\alpha)^2} + \int_0^t \frac{2kb(\tau)\,\mathrm{d}\tau}{(1-k\alpha(\tau))^3} \tag{23a}$$

$$T_k q = q(1-k\alpha)^2 \exp\left(\frac{\mathrm{i}kx}{1-k\alpha} + \int_0^t \frac{\mathrm{i}k^2 b(\tau)\,\mathrm{d}\tau}{(1-k\alpha(\tau))^2}\right) \tag{23b}$$

$$T_k t = t \qquad T_k R = R \tag{23c}$$

$$T_k a = \frac{a}{(1-k\alpha)^2} \tag{23d}$$

$$T_k b = \frac{b}{(1-k\alpha)^4} - \frac{a}{(1-k\alpha)^2}\int_0^t \frac{2kb(\tau)\,\mathrm{d}\tau}{(1-k\alpha(\tau))^3}. \tag{23e}$$

In (23) the parameter $k \in \mathbb{R}$.

Actually, one can prove that formulas (23) define a one-parameter *group* of transformations [10]. However, they are not *point* transformations: they act in a *non-local* way on variables a and b.

Transforming the bare 'linear problem' (3) according to the transformation (23) and then performing the gauge transformation given by

$$T_k \Psi = \exp\left(\frac{\mathrm{i}\sigma_3}{2}\left(\frac{kx}{1-k\alpha} + \int_0^t \frac{k^2 b(\tau)\,\mathrm{d}\tau}{(1-k\alpha(\tau))^2}\right)\right)\Psi \tag{24}$$

we obtain (now for *any* f linear in x!) the linear problem (16) with λ given by (17b).

Therefore extended Lie point symmetries isolate all f for which the NHNS system is known to be integrable. However we have not obtained any new integrable cases in this way.

7. Test for the Painlevé property of the NHNS system

The result that symmetry analysis did not provide new integrable cases, i.e. other than the known case $f = a(t)x + b(t)$, had been rather expected: it is believed that this is the only integrable case. The result of [24] confirms the supposition, namely it has been shown there that the inhomogeneity function f satisfying

$$f_{,xx} = 0 \tag{25}$$

is the only one for which the system (2) possesses the generalized Painlevé property, i.e. for which the general solution of that system extended to complex independent variables is free from branch and essential singularities whose position and shape depend on initial conditions [29]. The test for that property [29, 30] relies on constructing the explicit general solution in the form of a Laurent series about an arbitrary singularity surface. When a PDE has that property (is 'Painlevé integrable'), the recurrence formulae from which the coefficients are calculated, should be underdetermined so that they yield exactly $n-1$ arbitrary functions for a differential equation of nth order while the nth arbitrary function is the singularity surface itself. This test is not always equivalent to a check for complete integrability. Among others, the method does not prove convergence of the series, hence the positive result is not a sufficient condition. Besides, many examples of integrable equations may be given which possess solutions with movable branch points or essential singularities, even among the ODEs [31]. Nevertheless the test provides a useful hint for selection of integrable cases, especially when the tested equation contains arbitrary functions as parameters. Detailed discussion of the connection between the Painlevé property and integrability may be found in [32]. In appendix 3 we extend the result of [24] proving that the condition (25) is both necessary and sufficient for the 'Painlevé integrability' of equation (2).

8. Conclusions

The non-homogeneous, nonlinear Schrödinger system (2) (equivalent to the IHF equation) has been used as an example for testing a new integrability test which is based on Lie point symmetries [8, 12]. The result of this testing is positive in the sense that all cases selected by our test are integrable. Unfortunately, not all integrable cases can be isolated this way. We were able to work out an improved version of our test, based on a more general class of symmetries (extended Lie point symmetries [12]), which isolates all the integrable cases of (2). Thus, the test based on Lie *point* symmetries seems to select (in general) only *some* integrable cases. However, one can still hope to convert this test into a working *criterion* of integrability by admitting extended point symmetries. Obviously, tests of this kind require further examination. Some other examples are under consideration.

Acknowledgments

Sincere thanks are due to professor Decio Levi, Rome University 'La Sapienza', who was the first to observe that Lie point symmetries when applied to the NHNS system do not isolate all integrable cases.

Some results of this paper were presented at the 7th NEEDS workshop held in Gallipoli (Italy) in June 1991 [12]. One of the authors (JC) would like to thank its organizers for the invitation.

JC and AS were supported in part by the grant 566/2/91 GR 10 (KBN 2 0168 91 01). PG was supported in part by the grant PB 1274/P3/92/02 (KBN 2 2303 91 02).

Appendix 1. Lie point symmetries of the NHNS system

A1.1. Generators of Lie point symmetries of the NHNS system

Applying the definition (6) we obtain, in the standard way, that Lie point symmetries of the NHNS system are generated by vector fields of the form

$$w=(cx+h)\partial_x+\tau\partial_t+(\mathrm{i}kx+2\mathrm{i}\mu-c)q\partial_q-(\mathrm{i}kx+2\mathrm{i}\mu+c)\bar{q}\partial_{\bar{q}}+(\dot{\mu}-\dot{\tau}R)\partial_R \qquad (\mathrm{A}1.1a)$$

where $\mu=\mu(t)$ is an arbitrary function, while $c=c(t)$, $h=h(t)$, $\tau=\tau(t)$ and $k=$constant have to satisfy equations

$$\dot{c}x+\dot{h}=2kf \tag{A1.1b}$$

$$f_{,x}(cx+h)+f_{,t}\tau=(2c-\dot{\tau})f \tag{A1.1c}$$

where $f=f(x,t)$ is a coupling function of the NHNS system, and, finally, the dot denotes the derivative with respect to t.

A1.2. Lie point symmetries of the 'bare' linear problem

Lie point symmetries of the linear system (3) are generated by vector fields of the form (7*b*):

$$V=\xi\partial_x+\tau\partial_t+\eta\partial_q+\bar{\eta}\partial_{\bar{q}}+\gamma\partial_R+\begin{pmatrix} \mathrm{i}\mu & \nu \\ -\bar{\nu} & -\mathrm{i}\mu \end{pmatrix}\Psi\partial_\Psi \tag{A1.2}$$

where real ξ, τ, μ and complex η, ν are functions of x, t, q, $\bar{q}$, R. These functions have to satisfy the determining equations (8) which assume the following form:

$$D_x(\mu)=\mathrm{i}q\bar{\nu}-\mathrm{i}\bar{q}\nu+RD_x(\tau) \tag{A1.3a}$$

$$D_t(\mu)=-\bar{\nu}D_x(fq)-\nu D_x(f\bar{q})+\gamma+RD_t(\tau) \tag{A1.3b}$$

$$D_x(\nu)=-2\mathrm{i}\mu q+\eta+qD_x(\xi)+\mathrm{i}D_x(qf)D_x(\tau) \tag{A1.3c}$$

$$\begin{aligned} D_t(\nu)=2\mathrm{i}\nu R+2\mu D_x(qf)+\mathrm{i}D_x(q\xi f_{,x}+q\tau f_{,t})+\mathrm{i}D_x(f\eta)-\mathrm{i}D_x(fq)D_x(\xi)\\ -\mathrm{i}D_t(fq)D_x(\tau)+qD_t(\xi)+\mathrm{i}D_x(fq)D_t(\tau). \end{aligned} \tag{A1.3d}$$

The procedure of solving equations (A1.3) is simple and rather standard [11]. We obtain that infinitesimal Lie point symmetries of (3) are given by

$$V=(c_0x+h_0)\partial_x+\tau\partial_t+(2\mathrm{i}\mu-c_0)q\partial_q+(\dot{\mu}-\dot{\tau}R)\partial_R+\mu\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\Psi\partial_\Psi \tag{A1.4a}$$

where $\mu=\mu(t)$ is an arbitrary function, while the function $\tau=\tau(t)$ and constants c_0, h_0 have to satisfy the equation

$$f_{,x}(c_0x+h_0)+f_{,t}\tau=(2c_0-\dot{\tau})f. \tag{A1.4b}$$

Comparing the Lie algebras $\mathscr{A}$ and $\mathscr{A}'$ (see section 3) given by (A1.1) and (A1.4), respectively, one can easily see that they are identical iff $k=0$.

If the function f *is not linear* in x, then (A1.1*b*) implies $k=0$. Therefore in that case $\mathscr{A}=\mathscr{A}'$. The case of $f=a(t)x+b(t)$ has to be studied in more detail.

A1.3. Lie point symmetries of NHNS system in the case $f=ax+b$

Suppose that $f=a(t)x+b(t)$ and $f\not\equiv 0$. Both sides of the equations (A1.1*b*, *c*) are then linearin x. Therefore (A1.1*b*, *c*) become equivalent to the following system of four equations:

$$\dot{c}=2ka \tag{A1.5a}$$

$$\dot{h}=2kb \tag{A1.5b}$$

$$(a\tau)^{\cdot}=ac \tag{A1.5c}$$

$$ah+(b\tau)^{\cdot}=2bc. \tag{A1.5d}$$

From (A5a, b) we compute h and c:

$$h=2k\beta+h_1 \qquad \left(\beta:=\int_0^t b(\tau)\,\mathrm{d}\tau,\, h_1=\text{const}\right) \tag{A1.6a}$$

$$c=2k\alpha+c_1 \qquad \left(\alpha:=\int_0^t a(\tau)\,\mathrm{d}\tau,\, c_1=\text{constant}\right). \tag{A1.6b}$$

Then equations (A1.5c, d) assume the form

$$(a\tau)^{\cdot}=2ka\alpha+ac_1 \tag{A1.7a}$$

$$2k(a\beta-2b\alpha)+(b\tau)^{\cdot}+ah_1=2bc_1. \tag{A1.7b}$$

The equation (A1.7a) for $a\equiv 0$ becomes an identity. Then, it results from (A1.7b) that:

$$\tau=(2c_1\beta+d_1)/b \qquad \text{for } a\equiv 0 \text{ and } b\not\equiv 0 \tag{A1.8a}$$

where $d_1=$constant. If $a\not\equiv 0$, then we compute τ from (A1.7a)

$$\tau=(k\alpha^2+c_1\alpha+t_1)/a \qquad \text{for } a\not\equiv 0 \tag{A1.8b}$$

where d_1, t_1 are constants.

However, in the case $a\not\equiv 0$ the equation (A1.7b) still remains to be satisfied, giving the following constraint on a and b:

$$2k(a\beta-b\alpha)-bc_1+ah_1+(t_1+\alpha c_1+k\alpha^2)(b/a)^{\cdot}=0. \tag{A1.9}$$

Assuming that $\dot{\alpha}\equiv a$ has (locally) a constant sign we can determine the function $\beta=\beta(\alpha)$. Then $b=a\beta'$ and $(b/a)^{\cdot}=a\beta''$ (prime denotes the derivative with respect to α). Therefore:

$$(t_1+\alpha c_1+k\alpha^2)\beta''-(c_1+2k\alpha)\beta'+2k\beta+h_1=0. \tag{A1.10}$$

If $t_1=c_1=k=h_1=0$ then any β solves (A1.10). In other cases the solution to (A1.10) is given by

$$\beta=K_0+K_1\alpha+\tfrac{1}{2}K_2\alpha^2 \tag{A1.11a}$$

where K_j are constants subject to the constraint

$$2kK_0-c_1K_1+t_1K_2=h_1. \tag{A1.11b}$$

Differentiating (A1.11a) we obtain finally that the system (A1.5) has a non-trivial solution iff [26]

$$b(t)=K_1a(t)+K_2a(t)\int_0^t a(\tau)\,\mathrm{d}\tau \tag{A1.12}$$

or (taking into account earlier considerations) $a\equiv 0$.

For other a, b the only solution of (A1.5) is given by

$$c=h=k=\tau=0. \tag{A1.13}$$

The vector field w is parameterized in this case only by an arbitrary function $\mu=\mu(t)$.

Thus we have three different cases:

(i) $f=b(t)$
The algebra $\mathscr{A}$ is spanned by:

$$w_1=2\beta\partial_x+\mathrm{i}xq\partial_q \tag{A1.14a}$$

$$w_2=x\partial_x+(2\beta/b)\partial_t-q\partial_q-2(\beta/b)^{\cdot}R\partial_R \tag{A1.14b}$$

$$w_3=\partial_x \tag{A1.14c}$$

$$w_4=(1/b)\partial_t-(1/b)^{\cdot}R\partial_R \tag{A1.14d}$$

$$\omega_\mu=2\mathrm{i}\mu q\partial_q+\dot{\mu}\partial_R. \tag{A1.14e}$$

The commutators read as follows:

$$[w_2,w_1]=w_1 \qquad [w_3,w_1]=\tfrac{1}{2}\omega_{\mu\equiv1} \qquad [w_4,w_1]=2w_3 \qquad [w_3,w_2]=w_3$$

$$[w_4,w_2]=2w_4 \qquad [w_4,w_3]=0 \qquad [w_1,\omega_\mu]=0 \qquad [w_3,\omega_\mu]=0$$

$$[\omega_\mu,\omega_\nu]=0 \qquad [w_2,\omega_\mu]=2\omega_{\beta\dot{\mu}/b} \qquad [w_4,\omega_\mu]=\omega_{\dot{\mu}/b}.$$

The algebra $\mathscr{A}'$ is generated by $w_2, w_3, w_4, \omega_\mu$, i.e. $\mathscr{A}\neq\mathscr{A}'$.

(ii) $f=a(x+K_1+K_2\int a)$
The algebra $\mathscr{A}$ is spanned by

$$v_1=(2x+2K_1+K_2\alpha)a\partial_x+\frac{\alpha^2}{a}\partial_t+(\mathrm{i}x-2\alpha)q\partial_q-\left(\frac{\alpha^2}{a}\right)^{\cdot}R\partial_R \tag{A1.15a}$$

$$v_2=(x+K_1)\partial_x+(\alpha/a)\partial_t-q\partial_q-(\alpha/a)^{\cdot}R\partial_R \tag{A1.15b}$$

$$v_3=-K_2\partial_x+(1/a)\partial_t+(\dot{a}/a^2)R\partial_R \tag{A1.15c}$$

$$\omega_\mu=2\mathrm{i}\mu q\partial_q+\dot{\mu}\partial_R. \tag{A1.15d}$$

The commutators are given by:

$$[v_2,v_1]=v_1+\tfrac{1}{2}\omega_{\mu\equiv K_1} \qquad [v_3,v_1]=2v_2-\tfrac{1}{2}\omega_{\mu\equiv K_2} \qquad [v_3,v_2]=v_3$$

$$[\omega_\mu,\omega_\nu]=0 \qquad [v_1,\omega_\mu]=\omega_{\dot{\mu}\alpha^2/a} \qquad [v_2,\omega_\mu]=\omega_{\dot{\mu}\alpha/a}$$

$$[v_3,\omega_\mu]=\omega_{\dot{\mu}/a}.$$

The algebra $\mathscr{A}'$ is generated by v_2, v_3, ω_μ, i.e. $\mathscr{A}\neq\mathscr{A}'$.

(iii) Other f linear in x.
The algebras $\mathscr{A}$ and $\mathscr{A}'$ are identical and spanned by ω_μ (A1.15*d*).

Appendix 2. Extended Lie point symmetries of NHNS system

A2.1. Generators of extended point symmetries of NHNS system

Extended Lie point symmetries of the NHNS system (2) are generated by vector fields of the form (18):

$$\tilde{v}=\xi\partial_x+\tau\partial_t+\eta\partial_q+\bar{\eta}\partial_{\bar{q}}+\gamma\partial_R+\Phi\partial_f \tag{A2.1}$$

where generators ξ, τ, η, $\bar{\eta}$, γ depend on x, t, q, $\bar{q}$, R, and Φ is a function of x, t.

The determining equations read as follows (see (20*a*)):

$$0=\mathrm{i}D_t(\eta)-\mathrm{i}D_t(\xi)q_x-\mathrm{i}D_t(\tau)q_t+D_x^2(q\Phi+\eta f)-2D_x(\xi)D_x^2(fq)$$
$$-2D_x(\tau)D_xD_t(fq)-D_x^2(\xi)D_x(fq)-D_x^2(\tau)D_t(fq)+2\eta R+2q\gamma \qquad \text{(A2.2}a\text{)}$$

$$D_x(\gamma)-D_x(\xi)R_x-D_x(\tau)R_t$$
$$=\bar{q}f\{D_x(\eta)-D_x(\xi)q_x-D_x(\tau)q_t\}+qf\{D_x(\bar{\eta})-D_x(\xi)\bar{q}_x-D_x(\tau)\bar{q}_t\}$$
$$+2q\bar{q}\{D_x(\Phi)-D_x(\xi)f_{,x}-D_x(\tau)f_{,t}\}$$
$$+q_x\{\bar{\eta}f+\bar{q}\Phi\}+\bar{q}_x(\eta f+q\Phi\}+2f_{,x}\{\eta\bar{q}+\bar{\eta}q\} \qquad \text{(A2.2}b\text{)}$$

where, because of (2), q_t and R_x depend on other variables:

$$q_t=\mathrm{i}fq_{xx}+2\mathrm{i}f_{,x}q_x+\mathrm{i}f_{,xx}q+2\mathrm{i}qR \qquad \text{(A2.3}a\text{)}$$

$$R_x=fq_x\bar{q}+f\bar{q}_xq+2f_{,x}q\bar{q}. \qquad \text{(A2.3}b\text{)}$$

Equations (A2.2) are polynomials in derivatives. Assuming $f\not\equiv 0$ and equating to zero coefficients of R_t, q_{xt}, $\bar{q}_{xx}$, q_xq_{xx} and q_x^2 in (A2.2*a*), we obtain immediately that

$$\tau=\tau(t) \qquad \xi=\xi(x) \qquad \text{(A2.4}a\text{)}$$

and that η is linear in q and does not depend on R.

Now, one can easily see that $\mathrm{i}D_t(\eta)$, $-\mathrm{i}\dot{\tau}q_t$, $2\eta R$ and $2q\gamma$ are the only terms in (A2.2*a*) which depend on R. The first three of them are linear in R therefore γ has to be linear in R as well.

Equating to zero the coefficient of R in (A2.2*a*) we obtain that

$$\eta=A(x,t)q \qquad \gamma=-\dot{\tau}R+B(x,t,q,\bar{q}) \qquad \text{(A2.4}b\text{)}$$

where A is a complex function and B is real.

After substituting (A2.3) and (A2.4) the determining equations (A2.2) assume the following form:

$$(\Phi+f\dot{\tau}-2f\xi_{,x})q_{xx}+(2\Phi_{,x}+2f_{,x}\dot{\tau}-4f_{,x}\xi_{,x}-f\xi_{,xx}-\mathrm{i}\xi_{,t}+2fA_{,x})q_x$$
$$+(\Phi_{,xx}+f_{,xx}\dot{\tau}-2f_{,xx}\xi_{,x}-f_{,x}\xi_{,xx}+fA_{,xx}+2f_{,x}A_{,x}+\mathrm{i}A_{,t}+2B)q=0 \qquad \text{(A2.5}a\text{)}$$

$$B_{,q}q_x+B_{,\bar{q}}\bar{q}_x+B_{,x}$$
$$=(2\Phi_{,x}+2f_{,x}\dot{\tau}+f(A+\bar{A})_{,x}+2f_{,x}(A+\bar{A}))q\bar{q}$$
$$+(q_x\bar{q}+q\bar{q}_x)(\Phi+f(\dot{\tau}+A+\bar{A})). \qquad \text{(A2.5}b\text{)}$$

Equating to zero coefficients of q_{xx}, q_x, $\bar{q}_x$ we obtain a system of equations for functions ξ, τ, A, B and Φ. One can easily solve some of these equations to obtain

$$B=B(t) \qquad \text{(A2.6}a\text{)}$$

$$\Phi=f(2\xi_{,x}-\dot{\tau}). \qquad \text{(A2.6}b\text{)}$$

Moreover it is convenient to replace A by real functions c and s:

$$A=-c(x,t)+\mathrm{i}s(x,t). \qquad \text{(A2.6}c\text{)}$$

Now the system of equations resulting from (A2.5) assumes the following form:

$$\xi_{,t}=2fs_{,x} \qquad \text{(A2.7}a\text{)}$$

$$2fc_{,x}=3f\xi_{,xx} \qquad \text{(A2.7}b\text{)}$$

$$s_{,t}=2f\xi_{,xxx}+3f_{,x}\xi_{,xx}-fc_{,xx}-2f_{,x}c_{,x}+2B \tag{A2.7c}$$

$$c_{,t}=fs_{,xx}+2f_{,x}s_{,x} \tag{A2.7d}$$

$$2f\xi_{,xx}-fc_{,x}=2f_{,x}c-2f_{,x}\xi_{,x} \tag{A2.7e}$$

$$fc=f\xi_{,x}. \tag{A2.7f}$$

Assuming still that $f\not\equiv 0$ we have from (A2.7*b*, *f*) that

$$c=c(t) \qquad \text{and} \qquad \xi=cx+h \qquad \text{where } h=h(t) \tag{A2.8}$$

and equations (A2.7) become equivalent to the following system:

$$\dot{c}x+\dot{h}=2fs_{,x} \tag{A2.9a}$$

$$s_{,t}=2B \tag{A2.9b}$$

$$\dot{c}=fs_{,xx}+2f_{,x}s_{,x}. \tag{A2.9c}$$

Differentiating (A2.9*a*) and substituting this result into (A2.9*c*) we obtain that s is linear in x. Then, from (A2.9*b*) and (A2.6*a*) it follows that

$$s=kx+2\mu(t) \tag{A2.10a}$$

$$B=\dot{\mu} \tag{A2.10b}$$

where μ is a function and k is a constant. Thus the system (A2.9) becomes equivalent to the equation

$$\dot{c}x+\dot{h}=2kf. \tag{A2.11}$$

It is worthwhile pointing out that (A2.11) is identical with (A1.1*b*): the first condition defining standard Lie point symmetries. The second condition, (A1.1*c*), is an exemplification of (19).

The equation (A2.11) can be easily solved. Suppose first that $k=0$. Then, irrespective of f, we have

$$c=c_0=\text{constant} \qquad h=h_0=\text{constant}. \tag{A2.12}$$

Therefore, taking into account (A2.4), (A2.6) and (A2.10), we obtain:

$$\xi=c_0x+h_0 \qquad \tau=\tau(t) \tag{A2.13a}$$

$$\eta=(2\mathrm{i}\mu(t)-c_0)q \qquad \gamma=\dot{\mu}-\dot{\tau}R \tag{A2.13b}$$

$$\Phi=f(2c_0-\dot{\tau}). \tag{A2.13c}$$

If $k\neq 0$ then f has to be of the form $f=a(t)x+b(t)$, and we obtain that

$$c=2k\alpha+c_0 \qquad h=2k\beta+h_0 \tag{A2.14}$$

where α and β are defined by (12). Therefore

$$\xi=2k(\alpha x+\beta)+c_0x+h_0 \qquad \tau=\tau(t) \tag{A2.15a}$$

$$\eta=k(\mathrm{i}x-2\alpha)+2\mathrm{i}\mu(t)-c_0 \qquad \gamma=\dot{\mu}-\dot{\tau}R \tag{A2.15b}$$

$$\Phi=(ax+b)(4k\alpha+2c_0-\dot{\tau}). \tag{A2.15c}$$

The last formula implies that the set of functions f which are linear in x is transformed into itself. In this case it is natural to treat a and b rather than f as additional variables. The action of $\tilde{v}$ on a, b can be easily computed:

$$\Phi=\tilde{v}f=(\tilde{v}a)x+a(\tilde{v}x)+\tilde{v}b=(\tilde{v}a)x+a\xi+\tilde{v}b \tag{A2.16}$$

and, taking into account (A2.15a, c), we have

$$\tilde{v}a=(2m\alpha+c_0-\dot{\tau})a \tag{A2.17a}$$

$$\tilde{v}b=k(4\alpha b-2\beta a)+(2c_0-\dot{\tau})b-h_0a. \tag{A2.17b}$$

For $f=0$ the NHNS system reduces to a linear equation and this case is not considered in our paper. The determining equations (A2.2) then imply that $\Phi=0$, i.e. $f=0$ is transformed into $f=0$.

A2.2. Extended point symmetries of the 'bare' linear problem

Extended Lie point symmetries of the 'bare' linear problem (3) are generated by vector fields of the form:

$$\tilde{V}=\xi\partial_x+\tau\partial_t+\eta\partial_q+\bar{\eta}\partial_{\bar{q}}+\gamma\partial_R+\Phi\partial_f+\begin{pmatrix} \mathrm{i}\mu & \nu \\ -\bar{\nu} & -\mathrm{i}\mu \end{pmatrix}\Psi\partial_\Psi \tag{A2.18}$$

where ξ, τ, η, $\bar{\eta}$, γ, μ, ν are functions of x, t, q, $\bar{q}$, R and Φ is a function of x, t.

The determining equations read as follows (see (20b)):

$$D_x(\mu)=\mathrm{i}q\bar{\nu}-\mathrm{i}q\bar{\nu}+RD_x(\tau) \tag{A2.19a}$$

$$D_t(\mu)=-\bar{\nu}D_x(fq)-\nu D_x(f\bar{q})+\gamma+RD_t(\tau) \tag{A2.19b}$$

$$D_x(\nu)=-2\mathrm{i}\mu q+\eta+qD_x(\xi)+\mathrm{i}D_x(qf)D_x(\tau) \tag{A2.19c}$$

$$\begin{aligned} D_t(\nu)={}&2\mathrm{i}\nu R+2\mu D_x(qf)+\mathrm{i}D_x(\Phi q+f\eta)-\mathrm{i}D_x(fq)D_x(\xi)-\mathrm{i}D_t(fq)D_x(\tau)\\ &+qD_t(\xi)+\mathrm{i}D_x(fq)D_t(\tau). \end{aligned} \tag{A2.19d}$$

The general solution of (A2.19) can be obtained rather easily. We assume, similarly to section A1.1, that $f\neq 0$. Equating to zero the coefficient of q_x in (A2.19b) we have:

$$\nu=0. \tag{A2.20a}$$

Then, equating to zero coefficients of R_t, $\bar{q}_t$, q_x^2 and q_t in (A2.19d),

$$\xi=\xi(x,t) \qquad \tau=\tau(t). \tag{A2.20b}$$

Moreover, the coefficient of R_t in (A2.19c) implies that $\mu_R=0$.

Thus equations (A2.19a, b, c) assume the following form:

$$\mu=\mu(t) \qquad \eta=(2\mathrm{i}\mu-\xi_{,x})q \qquad \gamma=\dot{\mu}-\dot{\tau}R \tag{A2.21a}$$

and (A2.19d), after substituting (A2.21a), becomes equivalent to

$$\Phi=2f\xi_{,x}-f\dot{\tau} \qquad \text{and} \qquad f\xi_{,xx}=\mathrm{i}\xi_{,t}. \tag{A2.21b}$$

Finally, solving (A2.21b), we obtain that ξ, τ, η, γ, Φ are given exactly by (A2.13).

A2.3. The one-parameter group inserting the spectral parameter

Comparing the results of sections A2.1 and A2.2 we immediately obtain that in the case of the NHNS system algebras $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}'$ (defined by (20)) are not identical if and only if f is linear in x.

The vector field u corresponding to the parameter k, namely

$$u=2(x\alpha+\beta)\partial_x+(\mathrm{i}x-2\alpha)q\partial_q+2a\alpha\partial_a+(4b\alpha-2a\beta)\partial_b \tag{A2.22}$$

is an element of $\mathscr{A}$ but does not belong to $\mathscr{A}'$. Therefore the one-parameter group generated by u is expected to introduce the spectral parameter into the non-parametric linear problem (3).

We will denote the action of this group by subscript k (for example: $x_k:=T_kx$). Moreover

$$\alpha_k=\int_0^t a_k(\tau)\,\mathrm{d}\tau \qquad \beta_k=\int_0^t b_k(\tau)\,\mathrm{d}\tau. \tag{A2.23}$$

To compute the action of the one-parameter group generated by u one has to solve the following system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}k}(a_k)=2a_k\alpha_k \tag{A2.24a}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}(b_k)=4\alpha_k b_k-2a_k\beta_k \tag{A2.24b}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}(x_k)=2\alpha_k x_k+2b_k \tag{A2.24c}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}(q_k)=(\mathrm{i}x_k-2\alpha_k)q_k \tag{A2.24d}$$

assuming the following initial conditions:

$$a_0=a \qquad \alpha_0=\alpha \qquad b_0=0 \qquad \beta_0=\beta \qquad x_0=x \qquad q_0=q. \tag{A2.25}$$

First of all, let us compute a_k and α_k. They can be treated as functions of the group parameter k and t because t is an invariant of the considered group. Integrating (A2.24a) with respect to t we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}k}(\alpha_k)=\alpha_k^2. \tag{A2.26}$$

Solving (A2.26a) and then differentiating the result we have

$$\alpha_k=\frac{\alpha}{1-k\alpha} \qquad \text{and} \qquad a_k=\frac{a}{(1-k\alpha)^2}. \tag{A2.27}$$

Thus we have proved the formula (23d).

Now, taking into account (A2.27), one can easily check that equations (A2.24*b*, *c*, *d*) can be rewritten in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}k}((1-k\alpha)^4 b_k) = -2a(1-k\alpha)^2\beta_k \tag{A2.28a}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}((1-k\alpha)^2 x_k) = -2(1-k\alpha)^2\beta_k \tag{A2.28b}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}((1-k\alpha)^{-2} q_k) = ((1-k\alpha)^{-2} q_k)\mathrm{i}x_k. \tag{A2.28c}$$

It is convenient to introduce an auxiliary function $w_k = w_k(t)$:

$$w_k := 2(1-k\alpha)^{-2}\int_0^k [1-m\alpha(t)]^2\beta_m(t)\,\mathrm{d}m \tag{A2.29a}$$

which obviously satisfies the following initial conditions:

$$w_0(t)=0 \qquad w_k(0)=0. \tag{A2.29b}$$

Integrating both sides of (A2.28) with respect to k we have:

$$(1-k\alpha)^4 b_k = b - a(1-k\alpha)^2 w_k \tag{A2.30a}$$

$$(1-k\alpha)^2 x_k = x + (1-k\alpha)^2 w_k \tag{A2.30b}$$

$$(1-k\alpha)^{-2} q_k = q\exp\left(\mathrm{i}\int_0^k x_m\,\mathrm{d}m\right). \tag{A2.30c}$$

From (A2.29) we compute β_k and substitute it into (A2.30*a*) (taking into account that $b_k = \beta_{k,t}$). The result can be put in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}k}((1-k\alpha)^2 w_{k,t}) = 2(1-k\alpha)^{-2} b. \tag{A2.31}$$

Integrating (A2.31) twice we obtain w_k:

$$w_k = 2k\int_0^t [1-k\alpha(\tau)]^{-3} b(\tau)\,\mathrm{d}\tau. \tag{A2.32}$$

Therefore (A2.30*b*) becomes equivalent to (23*a*) while (A2.30*a*) becomes equivalent to (23*e*). Carrying out the integration in (A2.30*c*) we obtain exactly the formula (23*b*).

Thus we have proved that the vector field u (A2.22) generates the one-parameter group T_k given by (23).

Appendix 3. Test for the generalized Painlevé property of the NHNS system

This appendix contains the standard [29] test for the generalized Painlevé property of the system (2). We first perform it for a general inhomogeneity function $f(x, t)$: the result—linear dependence of the $f(x, t)$ on x,

$$f(x, t) = a(t)x + b(t) \tag{A3.1}$$

is already known from [24]. Then the test is continued to prove that (A3.1) is also the sufficient condition for that property.

Our system (2) extended to $(x, t) \in \mathbb{C}^2$ reads

$$iq_{,t} + (fq)_{,xx} + 2qR = 0 \tag{A3.2a}$$

$$-ip_{,t} + (fp)_{,xx} + 2pR = 0 \tag{A3.2b}$$

$$-R_{,x} + (fpq)_{,x} + f_{,x}pq = 0 \tag{A3.2c}$$

where p is the analytic extension of $\bar{q}$ to complex x and t.

According to [29] we look for solution q, p and R in the form of a Laurent series about some surface

$$\Phi(x, t) = 0 \tag{A3.3}$$

where Φ is analytic in x, t. Following [33] we exclude surfaces t=constant which are not suitable for the expansion as characteristics of the system (A3.2). For the remaining surfaces $\Phi_{,x}$ does not vanish identically so we may locally define the surface by (A3.3) solved with respect to x, assuming

$$\Phi(x, t) = x - \varphi(t) \tag{A3.4}$$

We now look for the solution of (A3.2) in the form of a Laurent series about the surface (A3.3) with Φ given by (A3.4), expanding also the inhomogeneity function f in a Taylor series about the same surface. Thanks to arbitrariness of the pole surface, we may assume that it does not coincide with the zero surface of f and begin the expansion of f with the zero-order term. The expansions will have the form

$$f(x, t) = \sum_{r=0}^{\infty} f_r(t)[x - \varphi(t)]^r \tag{A3.5a}$$

$$q(x, t) = \sum_{r=0}^{\infty} q_r(t)[x - \varphi(t)]^{r-\alpha} \tag{A3.5b}$$

$$p(x, t) = \sum_{r=0}^{\infty} p_r(t)[x - \varphi(t)]^{r-\beta} \tag{A3.5c}$$

$$R(x, t) = \sum_{r=0}^{\infty} R_r(t)[x - \varphi(t)]^{r-\gamma} \tag{A3.5d}$$

where the leading order exponents α, β, γ and coefficients q_0, p_0, R_0, obtained by substitution of the lowest-order terms for (A3.5) into (A3.2) read

$$\alpha = \beta = 1 \tag{A3.6a}$$

$$\gamma = 2 \tag{A3.6b}$$

$$R_0 = -f_0 \tag{A3.6c}$$

$$q_0 p_0 = -1. \tag{A3.6d}$$

One of the coefficients q_0, p_0 is an arbitrary function of time.

Substituting (A3.6*a*, *b*, *c*) into (A3.5) and then the solution of (A3.5) into (A3.2), we obtain a system of recurrence relations

$$n(n-3)f_0 q_n + 2q_0 R_n$$

$$= i(n-2)q_{n-1}\dot{\varphi} - i\dot{q}_{n-2} - (n-1)(n-2)\sum_{r=1}^{n}\frac{1}{r!}f_r q_{n-r} - 2\sum_{r=1}^{n-1} q_{n-r}R_r \tag{A3.7a}$$

$$n(n-3)f_0p_n+2p_0R_n$$
$$=-\mathrm{i}(n-2)p_{n-1}\dot{\varphi}+\mathrm{i}\dot{p}_{n-2}-(n-1)(n-2)\sum_{r=1}^{n}\frac{1}{r!}f_rp_{n-r}-2\sum_{r=1}^{n-1}p_{n-r}R_r \quad \text{(A3.7}b\text{)}$$

$$(n-2)f_0p_0q_n+(n-2)f_0q_0p_n-(n-2)R_n=-\sum_{m=0}^{n-1}\sum_{s=0}^{n-1}\frac{2n-m-s-2}{(n-m-s)!}f_{n-m-s}q_sp_m \quad \text{(A3.7}c\text{)}$$

in which the convention $f_r=p_r=q_r=0$ and $1/r!=0$ whenever $r<0$ has been used to simplify the notation.

For each $n\geqslant 0$ (A3.7) is a system of three linear equations with three unknowns. Its determinant reads

$$-f_0^2(n+1)n(n-2)(n-3)(n-4) \quad \text{(A3.8)}$$

The system is solvable iff it satisfies compatibility conditions at the 'resonances' $n=2$, $n=3$ and $n=4$.

The first resonance $n=0$ has already been considered in (A3.6*a*)–(A3.6*c*). The second one is due to the fact that the LHS of (A3.7*c*) is zero for $n=2$. After substitution of the zero-order coefficients (A3.6*a*)–(A3.6*c*), and the first-order coefficients

$$q_1=\mathrm{i}\dot{\varphi}q_0/(2f_0) \quad \text{(A3.9}a\text{)}$$

$$p_1=-\mathrm{i}\dot{\varphi}p_0/(2f_0) \quad \text{(A3.9}b\text{)}$$

$$R_1=0. \quad \text{(A3.9}c\text{)}$$

into (A3.7*c*), its RHS becomes $-f_2=-(\frac{1}{2})f_{,xx}$. Hence the necessary condition for 'Painlevé integrability' is the vanishing of $f_{,xx}$ [24], i.e. the linear dependence of f on x (A3.1).

We shall now prove that (A3.1) is sufficient for satisfying the other compatibility conditions.

So far, our procedure has produced two arbitrary functions—one is the shape of the singularity manifold $\Phi(x,t)$, the other is connected with the resonance at $n=0$; it may be taken, for example, as

$$C(t)=-\mathrm{i}q_0(t)=\mathrm{i}/p_0(t) \quad \text{(A3.10)}$$

according to (A3.6*d*). If the compatibility condition at $n=2$ is satisfied, we obtain the third arbitrary function at that resonance. We may assume, for example, R_2 as the arbitrary function of time

$$R_2(t)=D(t) \quad \text{(A3.11}a\text{)}$$

while q_2 and p_2 are calculated from (A3.7*a*, *b*) in terms of that function

$$p_2=(2\mathrm{i}CD-\dot{C})/(2C^2f_0) \quad \text{(A3.11}b\text{)}$$

$$q_2=(2\mathrm{i}CD-\dot{C})/(2f_0). \quad \text{(A3.11}c\text{)}$$

At $n=3$ the equations (A3.7*a*) and (A3.7*b*) are linearly dependent (both sides). This yields another arbitrary function of time $E(t)$. For symmetry of the formulae for q_3, p_3 we have chosen this so that

$$q_3(1-E)/C-p_3EC=0. \quad \text{(A3.12)}$$

Then

$$q_3 = UE \tag{A3.13a}$$

$$p_3 = U(E-1)/C^2 \tag{A3.13b}$$

$$R_3 = (C\varphi f_0 - 4CDf_0 f_1 - 2\mathrm{i}\dot{C} f_0 f_1 - C\dot{\varphi} f_0)/(4Cf_0^2) - f_3. \tag{A3.13c}$$

where

$$U = (-\mathrm{i}C\varphi f_0 - 2\mathrm{i}C\dot{\varphi}^2 f_1 - 12\mathrm{i}CDf_0 f_1 + 6\dot{C} f_0 f_1 - 12\mathrm{i}Cf_0^2 f_3 + \mathrm{i}C\dot{\varphi}\dot{f}_0)/(4f_0^3) \tag{A3.14}$$

The last resonance is connected with linear dependence of the coefficients at the first power of Φ, namely the LHS of (A3.7*a*) multiplied by p_0 plus the LHS of (A3.7*b*) multiplied by q_0 yields the double LHS of (A3.7*c*). Substitution of the actual values of the coefficients $q_0, \ldots, q_3, p_0, \ldots, p_3, R_0, \ldots, R_3$, i.e. (A3.10) and (A3.6*c*) for the zero order, (A3.9*a*, *b*, *c*) for the first order, (A3.11*a*, *b*, *c*) for the second and (A3.13*a*, *b*, *c*) for the third, yields—under the condition $f_2 = 0$—expressions having the same linear dependence also on the RHS of (A3.7*a*, *b*, *c*). Thus, the assumption about the shape of the inhomogeneity function f (A3.1) is both the necessary and sufficient condition for the Painlevé property of the system (2).

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